UCLouvain

LACTU 2170
Homeworks 25-26 Part 2

Rapport

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Introduction

For this first homework, we are working for a French insurance company and we are asked to evaluate the price of variables annuities. We will then implement methods in order to compute their values.

We are going to dig into 3 methods, the first one will be the analytical solution, the second one will be a valuation by tree and thirdly it will be with Monte Carlo. For each method, we will analyze the convergence of the method to the analytical solution.

Then we will analyze the results themselves, by plotting the graphs of the price and its components versus the age of the assured, the volatility and the guaranteed rate. This will help us understand the sensitivity of the product to these parameters.

Question 1 — Valuation of the contract and derivation of the analytical solution

First, we assume that $(S_t)_{t>=0}$ is the market value of an indice, it is a random process which is driven by a geometric Brownian motion of volatility $\sigma = 12\%$ and initial value $S_0 = 100$. Through this computation we will often use the risk free rate and we assume that it is constant and equal to $r = 3.2\%$. We are asked to price a product which intend to deliver a payoff of

$$C_0 \max(\frac{S_T}{S_0}, e^{gT}) \tag{1}$$

at maturity $T = 5$ with $g = 2.5\%$.

In case of premature death at time t then the inheritor will receives

$$C_0 \max(\frac{S_t}{S_0}, e^{gt}) \tag{2}$$

We already can identify that we need to price a combinaison of GMDB and GMDA as we guarantee a minimum accumulation benefit independently of the performance of the asset S_t. The GMDB come from the fact that we guarantee a minimum capital in case of death of the client.

From the valuation of variables annuities using tree we can retrieve the fact that the present value of an European option with payoff F_T at maturity T is given by :

$$F_0 = \mathbb{E}^Q[e^{-rT} \times F_T] \tag{3}$$

But reformulating 1 as follows:

$$\begin{aligned}
C_0 \max\left(\frac{S_T}{S_0}, e^{gT}\right) &= \frac{C_0}{S_0} \max(S_T, S_0 e^{gT}) \\
&= \frac{C_0}{S_0} \left[S_0 e^{gT} + (S_T - S_0 e^{gT})_+\right]
\end{aligned}$$

We identify a Call option with strike $K = S_0 e^{gT}$. Injecting into (3):

$$F_0 = \mathbb{E}^Q\left[e^{-rT} \cdot \frac{C_0}{S_0} \left(S_0 e^{gT} + (S_T - S_0 e^{gT})_+\right)\right] \tag{4}$$

$$= \frac{C_0}{S_0} \left[e^{(g-r)T} S_0 + \mathbb{E}^Q\left[e^{-rT}(S_T - S_0 e^{gT})_+\right]\right] \tag{5}$$

But from BS, we know that :

$$C = \mathbb{E}^Q\left[e^{-r(T-t)}(S_T - K)_+ \mid \mathcal{F}_t\right] = S_t \Phi(d_1^{BS}) - K e^{-r(T-t)} \Phi(d_2^{BS}) \tag{6}$$

$$P = \mathbb{E}^Q\left[e^{-r(T-t)}(K - S_T)_+ \mid \mathcal{F}_t\right] = K e^{-r(T-t)} \Phi(-d_2^{BS}) - S_t \Phi(-d_1^{BS}) \tag{7}$$

where

$$\begin{aligned}
d_1^{BS} &= \frac{\ln(S_t/K) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} \\
d_2^{BS} &= d_1^{BS} - \sigma\sqrt{T-t}
\end{aligned}$$

and $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal $\mathcal{N}(0,1)$. We can then infer this term into the previous equation (4) with $t = 0$.

To end the computation we still need to multiply by the probability of survival of the customer which is given by ${}_Tp_x$ as it is the probability that the assured person live till the maturity T while we know that he is currently x years old.

We finally get :

Result Valuation GMAB

$$F_0^{GMAB} = {}_Tp_x \frac{C_0}{S_0}\left[e^{(g-r)T}S_0 + S_0\Phi(d_1^{BS}) - S_0e^{T(g-r)}\Phi(d_2^{BS})\right] \tag{8}$$

$$= {}_Tp_x C_0\left[e^{(g-r)T} + \Phi(d_1^{BS}) - e^{T(g-r)}\Phi(d_2^{BS})\right] \tag{9}$$

Now for the GMDB, we are going to proceed in the same fashion but we have to be careful because here we are going to sum up all the cases possible. The case that the customer die in the first,second,third,... year. We need to develop the following term :

$$\begin{aligned}\mathbb{P}(k-1 \le \tau \le k) &= \mathbb{P}(k-1 \le \tau)\mathbb{P}(\tau \le k|\tau \ge k-1)\\ &= {}_{k-1}p_x \times q_{x+k-1}\end{aligned}$$

Intuitively, we are computing the probability that the remaining lifetime of the person is between k and $k-1$. Which is also equal to the probability of living more than $k-1$ year times the probability of living less than k knowing that we have already lived $k-1$ year.

We then value the payoff for each possible year of death $k = 1, \ldots, T$, weighted by the probability of dying in that year. Reformulating the death payoff as follows:

$$\begin{aligned}C_0 \max\left(\frac{S_k}{S_0}, e^{gk}\right) &= \frac{C_0}{S_0}\max(S_k, S_0e^{gk})\\ &= \frac{C_0}{S_0}\left[S_0e^{gk} + (S_k - S_0e^{gk})_+\right]\end{aligned}$$

We identify again a Call option but now with strike $K_k = S_0e^{gk}$ and maturity k. Injecting into (3) and summing over all possible years of death:

$$\begin{aligned}F_0^{GMDB} &= \sum_{k=1}^{T} {}_{k-1}p_x \cdot q_{x+k-1} \cdot \mathbb{E}^Q\left[e^{-rk} \cdot \frac{C_0}{S_0}\left(S_0e^{gk} + (S_k - S_0e^{gk})_+\right)\right]\\ &= \sum_{k=1}^{T} {}_{k-1}p_x \cdot q_{x+k-1} \cdot \frac{C_0}{S_0}\left[e^{(g-r)k}S_0 + \mathbb{E}^Q\left[e^{-rk}(S_k - S_0e^{gk})_+\right]\right]\end{aligned}$$

Applying the Black-Scholes formula with $t = 0$, $S_t = S_0$ and $K_k = S_0e^{gk}$ for each maturity k:

$$\begin{aligned}d_1^{BS}(k) &= \frac{\ln(S_0/K_k) + \left(r + \frac{\sigma^2}{2}\right)k}{\sigma\sqrt{k}} = \frac{\left(r - g + \frac{\sigma^2}{2}\right)k}{\sigma\sqrt{k}}\\ d_2^{BS}(k) &= d_1^{BS}(k) - \sigma\sqrt{k}\end{aligned}$$

Result Valuation GMDB

$$
\begin{aligned}
F_0^{GMDB} &= \sum_{k=1}^{T} {}_{k-1}p_x \cdot q_{x+k-1} \cdot \frac{C_0}{S_0} \left[e^{(g-r)k} S_0 + S_0 \Phi(d_1^{BS}(k)) - S_0 e^{(g-r)k} \Phi(d_2^{BS}(k)) \right] \\
&= \sum_{k=1}^{T} {}_{k-1}p_x \cdot q_{x+k-1} \cdot C_0 \left[e^{(g-r)k} \left(1 - \Phi(d_2^{BS}(k))\right) + \Phi(d_1^{BS}(k)) \right] \qquad (10)
\end{aligned}
$$

The fair value of the full contract is therefore the sum of the GMAB and GMDB components:

Total Fair Value

$$
F_0 = F_0^{GMAB} + F_0^{GMDB} \qquad (11)
$$

Question 2 — Valuation of the contract through a binomial tree

The principle of the binomial tree is based on the risk neutral portfolio hedging and the discretization of the time, such that at each step, each node is split into two branches, one representing the possibility of the asset going up and one for the asset going down. We can then compute the value of the product at every timestep. This is thus an exhaustive method over a simplified model, as we compute every possible path of our discretized process (in contrast with Monte Carlo, see qustion 3).

The ingeniosity of this method resides in the fact that at first glance, you think that since we split a node into two branches at each time step, we have $O(2^n)$ possible paths, thus exponential complexity. But in reality, those branches recombine, and we have only $O(n)$ nodes at the last layer of the tree, each weighted by a binomial probability. As we will see in the analysis of computation time, this ensure our method is usable in practice.

To implement this method we need to recall equation (3). In the case of a pricing by tree, the price of our product at time $t = 0$ is given by

$$F = \mathbb{E}^{\mathbb{Q}}\big(e^{-rT} F_T\big) = e^{-rT} \sum_{j=0}^{n} \binom{n}{j} p^j (1-p)^{n-j} \, f_{u^j d^{n-j}}.$$

with :

$$u = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}\right], \qquad d = \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta - \sigma\sqrt{\Delta}\right], \qquad p = \frac{e^{r\Delta} - d}{u - d}.$$

The payoff is computed such that $f_{u^j d^{n-j}}$ is computed with $S_T = S_0 \, u^j d^{n-j}$.

In our case we have implemented the algorithm as follow :

Algorithm 1 PricingTree

Require: S_0, r, σ, g, n, t
Ensure: F_0

1: $K \leftarrow S_0 \, e^{g\,t}$ ▷ Strike forward
2: $\Delta \leftarrow t \,/\, n$
3: $u \leftarrow \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta + \sigma\sqrt{\Delta}\right]$
4: $d \leftarrow \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta - \sigma\sqrt{\Delta}\right]$
5: $p \leftarrow \dfrac{e^{r\Delta} - d}{u - d}$
6: somme $\leftarrow 0$
7: **for** $j = 0, 1, \ldots, n$ **do**
8: $\quad S_T \leftarrow S_0 \cdot u^j \cdot d^{n-j}$
9: $\quad$ payoff $\leftarrow \max(S_T - K,\ 0)$
10: $\quad$ somme $+= \binom{n}{j} p^j (1-p)^{n-j} \cdot$ payoff
11: **end for**
12: $F \leftarrow e^{-r\,t} \cdot$ somme **return** F

A higher number of steps will lead to a more accurate approximation, but of course will exponentially inscrease the number of nodes, and thus the computation time. For the following graphs, the age of the assured was of $x = 40$ years.

Figure 1 shows a linear convergence of the price to the analytical solution as a function of the number of steps n. This is consistent with the fact that the binomial tree is a first-order method.

Figure 2 confirms the linear scaling: on a log-log plot the computation time lies parallel to the $O(n)$ reference line. This is what we expected in the explanation of the method, as we have only $O(n)$ nodes at each layer of the tree. Disclaimer, GMDB is sometimes more expensive to compute than Fair value on the graph, this is because we time each of the calculations individually, and since computation time has some variability, sometimes GMDB is faster than Fair value, which uses GMDB as a component, but this is of course only due to the variability of the computation time and not to the complexity of the method.

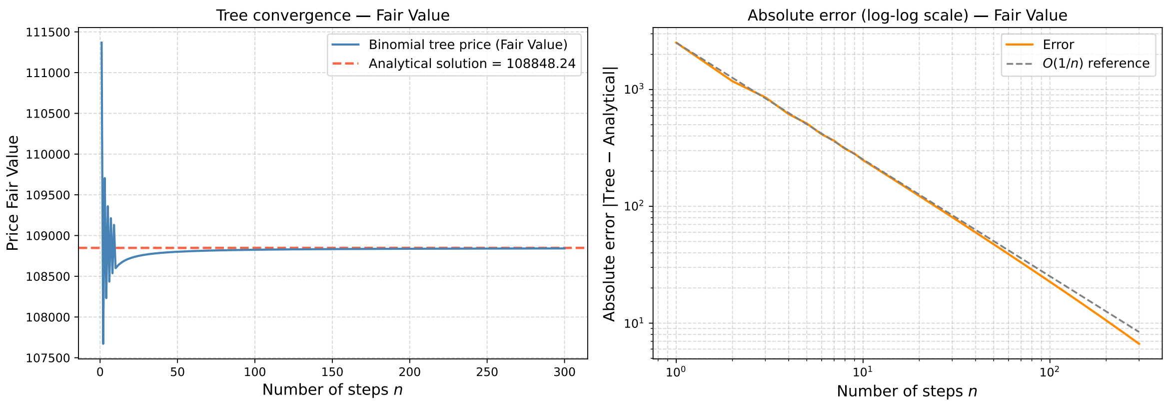


Figure 1: Convergence of the binomial-tree price as a function of the number of time steps n.

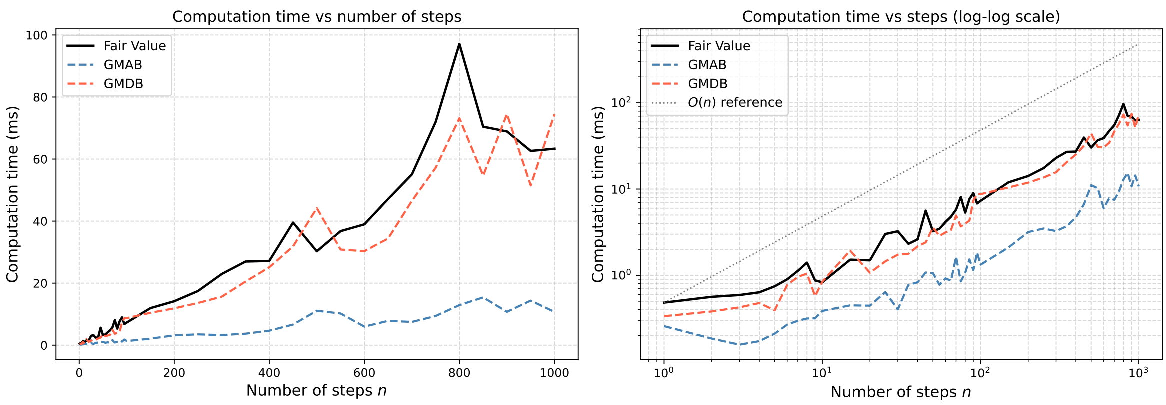
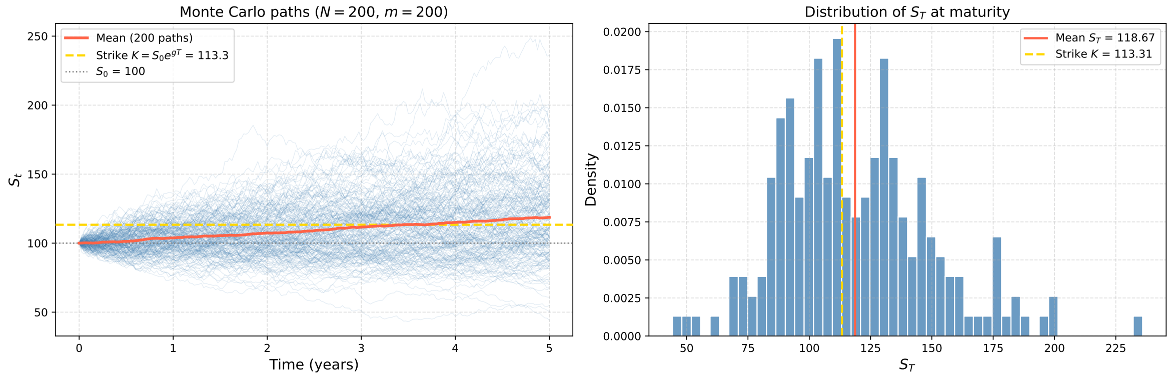


Figure 2: Computation time of the binomial-tree pricer as a function of the number of steps n. The right panel uses a log-log scale with an $O(n)$ reference line, confirming the linear complexity of the direct formula.

Question 3 — Valuation of the contract through Monte Carlo

In opposition to the binomial tree, Monte Carlo does not do an exhaustive search of paths of a simplified (to binary) process, but rather simulates a large number of paths, still on the discretized time, but using a brownian motion to simulate the stock price rather than a binary probability of going up or down. Then it takes the average of the simulations. The graph below gives a feeling of the simulation of multiple trajectories.



The algorithm we implemented is the following:

Algorithm 2 MONTECARLOPRICING

Require: S_0, r, σ, g, T, N, m
Ensure: F_0

$K \leftarrow S_0\, e^{g\,T}$ ▷ Forward strike
$\Delta t \leftarrow T \,/\, m$
Initialise matrix $\mathbf{S} \in \mathbb{R}^{N\times(m+1)}$ with $\mathbf{S}_{:,0} = S_0$
for $j = 1, 2, \ldots, m$ **do**
  Draw $Z_1, \ldots, Z_N \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$
  $\mathbf{S}_{:,j} \leftarrow \mathbf{S}_{:,j-1} + r\,\mathbf{S}_{:,j-1}\,\Delta t + \sigma\,\mathbf{S}_{:,j-1}\,\sqrt{\Delta t}\cdot \mathbf{Z}$
end for
for $i = 1, 2, \ldots, N$ **do**
  $\text{payoff}_i \leftarrow \max(S_{i,m} - K,\ 0)$
end for
$F_0 \leftarrow e^{-rT} \cdot \dfrac{1}{N} \sum_{i=1}^{N} \text{payoff}_i$ **return** F_0

Here we have two parameters. As before, we can play with the number of time steps m, but in addition, we can also change the number of simulations N. Increasing each of these parameters will increase the precision of our estimation, but also the computation time.

On figure 3 we can see the Monte Carlo method converges to the analytical solution. We tested the impact of the parameter N on the convergence, and found the error was a function of $O(1/\sqrt{N})$ as expected per the central limit theorem.

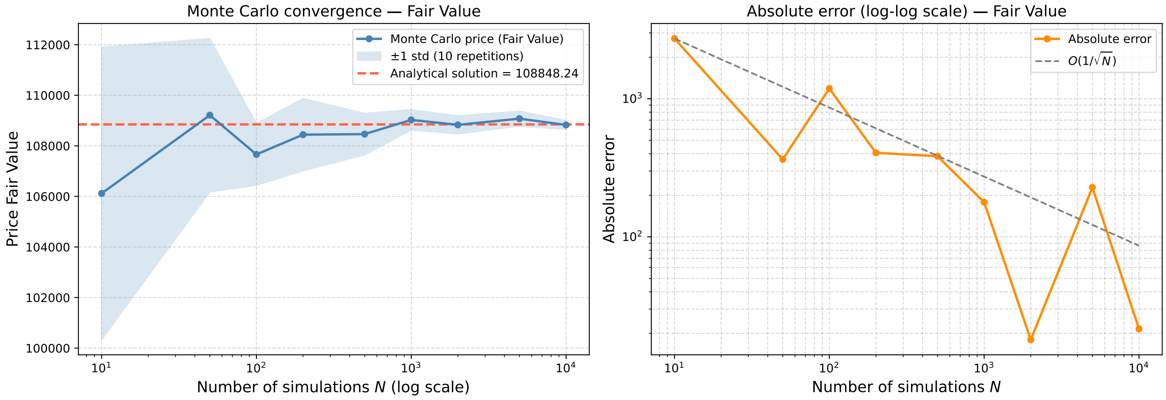


Figure 3: Convergence of the Monte Carlo price as a function of the number of time steps n and the number of simulations m.

Analysis of the results

Now that we have tested various methods, let's look at the results themselves. More specifically, let's analyze the sensitivity of the price to the age, the volatility, and the guaranteed rate.

By looking at figure 4, the first thing that we can connfirm is the fact that the price of the GMDB increase with the age of the assured, and inversely for the GMAB.

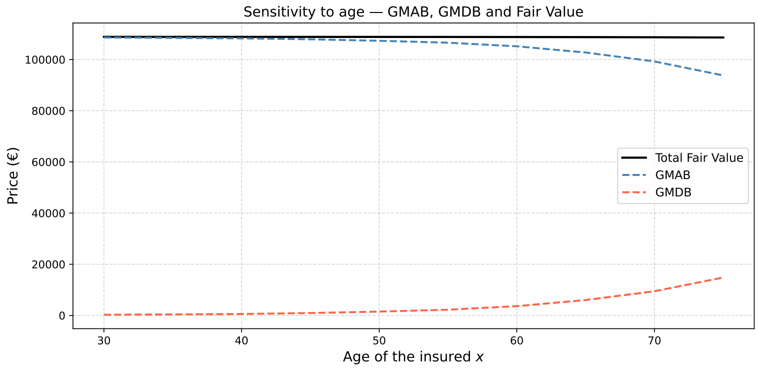


Figure 4: Sensitivity of the price to the age of the customer.

The second intuition we want to confirm is the fact that a greater volatility should imply a greater price, as the payoff of the call option increases with the volatility. This is what we can see on figure 5. Please note that the two following graphs project a young client of 40 years old, the GMDB is thus really low.

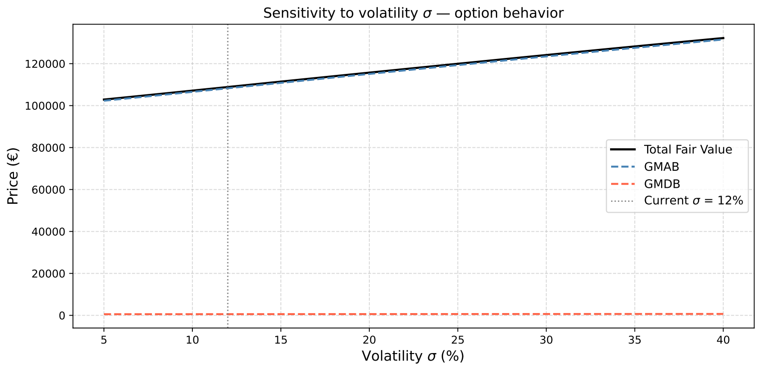


Figure 5: Sensitivity of the price to the volatility σ.

Finally, of course, if the guaranteed rate g increases, the price of the product should also increase as the payoff of the call option increases with the strike. This is what we can see on figure 6.

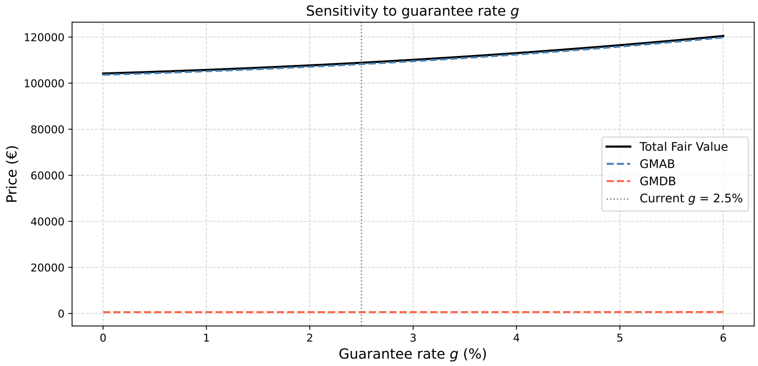


Figure 6: Sensitivity of the price to the guaranteed rate g.